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# Max-fully cancellation modules

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**Abstract:-**

Let R be a commutative ring with identity and let M be a unital an R`-module. We introduce the concept of max-fully cancellation R-module , where an R-module M is called max-fully cancellation if for every nonzero maximal ideal I of R and every two submodules$N_1$ and$N_2$ , of M such that $IN_1$=$IN_2$ , implies $N_1 = N_2$ . some characterization of this concept is given and some properties of this concept are proved. The direct sum and the trace of module with max-fully cancellation modules are studied , also the localization of max-fully cancellation module are discussed..

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## INTRODUCTION:-

Throughout this thesis all rings are commutative rings with unity and all modules are unital modules. Gilmer in [14] introduced the concept of cancellation ideal , where an ideal I of a ring R is said to be cancellation if whenever AI=BI with A and B are ideals of R , implies A=B . Also ,D.D. Anderson and D.F. Anderson in[3] ,studied the concept of cancellation ideals . In 1992 A.S. Mijbass in [16], gave the generalization of this concept namely cancellation module (weakly cancellation module) , where an R-module M is called cancellation ( weakly cancellation ) if wherever I and J are two ideals of R , with IM=JM implies I=J (I+M =J+M)

Inaam, M .A .Hadi, A.A. Elewi in [5], introduced the concept of fully cancellation module , where an R-module M is called fully cancellation module if for each ideal I of R and for each submodules $N_1, N_2$ , of M such that $IN_1=IN_2$ , implies $N_1 = N_2$ .

In section One , we introduce the definition of max-fully cancellation module and we give some characterizations for a module to be max-fully cancellation module , see proposition(1.7), also many propositions and results related with this concept are given.

In section two, we study the direct sum of max-fully cancellation modules and many of important results are given , see proposition (2.2) , proposition(2.7) and proposition(2.8).

In section three , we study the behavior of max-fully cancellation modules under localization . we show that : M is max-fully cancellation R-module if and only if M is locally max-fully cancellation, see proposition (3.5).

In section four , we discuss the relationship between max-fully cancellation module and its trace T(M). However in class of multiplication and projective module we give a condition on T(M) under which M be max-fully cancellation module , see proposition (4.10),also we prove that the max-fully cancellation module and its trace are equivalent under certain condition , see proposition (4.11)

## §1 max-fully cancellation modules

In this section ,we introduce the concept of max-fully cancellation module as a generalization of fully cancellation module .we give some characterizations and establish some basic properties of this concept .

We introduce the following definition

### Definition(1.1)[4]:-

Let $I$ be a proper ideal of a ring .Then $I$ is said to be **maximal ideal** of R , if there exists an ideal $J$ of R such that $I \subsetneq J \subseteq R$ then $J = R$.

### proposition (1.2)[4] *:-*

**(1)** Every proper ideal is contained in a maximal ideal .

**(2)** Every commutative ring with identity contains maximal ideal.

### Definition(1.3):-

An R-module M is called **faithful** , if $\text{ann}_R$(M)=0,where $\text{ann}_R$(M)={r∈ $R$:rm=0 ∀m∈ $M$} .

### Definitio(1.4)[15]:-

An R –module M is called **cancellation R-module**, if AM=BM ,where A and B two ideals of R ,then A=B

### Proposition(1.5) [15]:-

**(1)** Every cancellation R-module is faithful .

**(2)** If M is multiplication faithful finite generated Then M is cancellation.

### Defi1nition (1.6):-

An R-module M is called **max-fully cancellation module** if for every non zero maximal ideal Iof R and for every submodules $N_1$ and $N_2$ of M such that $IN_1 = IN_2$ ,then $N_1 = N_2$**.**

### Remarks and Examples (1.7):-

**(1)** Z as a Z-module is max-fully cancellation module .

Since if we take I=pZ ;p is prime number

also , $N_1=< x_1 >$and $N_2 =< x_2 >$where $x_1 ,x_2 \in \mathbb{Z}$ .

Assume that $IN_1 = IN_2$ , then$px_1Z =px_2Z$ .

and hence p $x_1$ =p $x_2$a and $px_1 =pm_1b$ ,where a ,b $\in \mathbb{Z}$.

Therefore $px_1 =px_1ab$ ,then ab =1 and hence either ab=1 or ab =-1 .

In each case we get $px_1 =px_2$ which implies $x_1=x_2$ and hence $N_1 =N_2$.

**2)** The Z-module $Z_6$ is not max-fully cancellation

Since ,if we take I=2Z ,$N_1=(\bar{2})$and $N_2 =Z_6$ .

Then (2Z) $(\bar{2})$ =(2Z) $Z_6$ .But$(\bar{2}) \neq Z_6$

(3) every fully cancellation R-module is max-fully cancellation R-module .But the converse is not true in general .

## For examples :

Consider $(\bar{3})$ as an R-module and R=$Z_{24}$ .

Then$(\bar{3})$ is max-fully cancellation R-module .

Since , $(\bar{2})$ is maximal ideal of R and$(\bar{9})$ , $(\overline{21})$ are two

submodules of $(\bar{3})$ such that $(\bar{2})$ $(\bar{9})$= $(\bar{2})$ $(\overline{21})$= $(\overline{18})$ .

Then$(\bar{9})$ =$(\overline{21})$ .

But it is not fully cancellation R-module .since , $(\bar{8})$ is an ideal of R and $(\bar{3})$ , $(\bar{0})$ are two submodules of$(\bar{3})$ such that $(\bar{8})$ $(\bar{3})$ =$(\bar{8})$ $(\bar{0})$= $(\bar{0})$ ,but $(\bar{3}) \neq (\bar{0})$ .

**(4)** The Z-module $Z_p^{\infty}$is not max-fully cancellation module.

Since , $Q_p$ =$\{\frac{m}{n}$ ,g.c.d(m ,n)=1 ;n=$p^i$ ,i=1,2,3,…….} is a submodule of $Q$ containing Z .

also , $Z_p^{\infty}$= $Q_p$ /Z ={x$\in Q$ ; x=$\frac{m}{P^i}$ +Z ;m$\in Z$ ,i=1,2,3,……}.

Let (P) be a maximal ideal of Z and $(\frac{1}{p}$ +Z) ,(0) be two submodules of $Z_p^{\infty}$, then we have (P) $(\frac{1}{p}$ +Z) = (P) (0) ,

But $(\frac{1}{p}$ +Z)$\neq$ (0).

**(5)**$Z_{12}$is not max-fully cancellation $Z_{12}$-module .

Let$(\bar{6})$ , $(\bar{0})$ be two submodules of $Z_{12}$ and $(\bar{2})$ be maximal ideal of $Z_{12}$ .

Since $(\bar{2})$ $(\bar{6})$ = $(\bar{2})$ $(\bar{0})$ =$(\bar{0})$ , But $(\bar{6}) \neq (\bar{0})$.

**(6)** The homomorphic image of the max-fully cancellation need not be max-fully cancellation module ,for example :-

We have from (1) that the Z-module Z is max-fully cancellation module .But Z/6Z$\cong Z_6$ is not max-fully-cancellation Z-module by (2) .

**(7)** Every submodule N of max-fully cancellation R-module M is also max-fully cancellation .

**Proof:-**

Let $I$ be a non zero maximal ideal of R such that$IN_1= IN_2$ .

where $N_1$ ,$N_2$ are any two submodules of N ,since $N_1$, $N_2$ are submodules of M and M is max-fully cancellation module ,then $N_1=N_2$ .

$which$ implies that N is max-fully cancellation .

As an application of (7) ,we get the following results in (8) and (9) .

**(8)**The intersection of two R-submodules of M which are at least one of them is max-fully cancellation R-submodule is also ,max-fully cancellation .

**Proof:-**

Let $N_1$ and $N_2$ be two submodules of an R-module M .It is known that $N_1 \cap N_2 \subseteq N_1$ .

Also $N_1 \cap N_2 \subseteq N_2$ ,so according to (7) , $N_1 \cap N_2$ is max-fully cancellation .

As a generalization of (8) ,we get:

**(9)**If $\{N_k\}_{k=1}^{n}$ is a finite collection of a submodules of an R-module M and $N_k$ is max-fully cancellation submodule for some k ,Then $\cap_{k=1}^{n} N_k$ is also max-fully cancellation .

## Proof:-

The proof is by induction on n .

The following theorem is a characterization of max-fully cancellation modules .

## Theorem(1.8):-

Let M be an R-module ,let $N_1$ ,$N_2$ are two submodule of M ,let $I$ be a non zero maximal ideal of R Then the following statement are equivalent

(1) M is max-fully cancellation module .

(2) if $IN_1 \subseteq IN_2$ then $N_1 \subseteq N_2$ .

(3) if $I \prec a \succ \subseteq IN_2$ then $a \in N_2$ where $a \in M$.

(4) $(IN_1 :_R IN_2) = (N_1 :_R N_2)$ .

## Proof :-

$(1) \Rightarrow (2)$.If $IN_1 \subseteq IN_2$ then $IN_2 = IN_1 + IN_2$ Which Implies

$IN_2 = I(N_1 + N_2)$, But M is max-fully cancellation module, then

$N_2 = (N_1 + N_2)$ and hence $N_1 \subseteq N_2$

$\Rightarrow (3)(1)$ : if $I \prec a \succ \subseteq IN_2$ then $\prec a \succ \subseteq N_2$ by (2)

Which implies ,$a \in N_2$ .

$(3) \Rightarrow (1)$: If $IN_1 = IN_2$ ,To prove that $N_1 = N_2$.

Let $a \in N_1$ then $I \prec a \succ \subseteq IN_1 \subseteq IN_2$ .

And hence $a \in N_2$ by (3)

Similarly ,we can show $N_2 \subseteq N_1$ .

Thus $N_1 = N_2$ .

$(1) \Rightarrow (4)$: let $r \in (IN_1 :_R IN_2)$ .Then $r\, IN_2 \subseteq IN_1$

So , $IrN_2 \subseteq IN_1$ and since (1) implies (2) , we have

$rN_2 \subseteq N_2$ .

Thus $r \in (N_1 :_R N_2)$ and hence $(IN_1 :_R IN_2) \subseteq (N_1 :_R N_2)$.

Let $r \in (N_1 :_R N_2)$. Then $rN_2 \subseteq N_1$ which implies $IrN_2 \subseteq IN_1$ .

And hence $rIN_2 \subseteq IN_1$ . Therefore $r \in (IN_1 :_R IN_2)$ and hence $(N_1 :_R N_2) \subseteq (IN_1 :_R IN_2)$

Then we get $(N_1 :_R N_2) = (IN_1 :_R IN_2)$ .

$(4) \Rightarrow (1)$: Let $IN_1 = IN_2$ Then by (4) $(IN_1 :_R IN_2) = (N_1 :_R N_2)$ .

But $(IN_1 :_R IN_2) = R$ (since $IN_1 = IN_2$ ).

Then $(N_1 :_R N_2) = R$. so $N_2 \subseteq N_1$ .

Similarly $(IN_2 :_R IN_1) = (N_2 :_R N_1)$ .Thus $(N_2 :_R N_1) = R$.

Which implies $N_1 \subseteq N_2$. Therefore $N_1 = N_2$ .

## Definition(1.9)[10]:-

An ideal $I$ of a ring R is called **cancellation ideal** if $AI = BI$ ,then A=B ,where A and B are two ideals of R ."

## Proposition(1.10):-

Let M be a max fully cancellation R-module . If M is a cancellation module ,then every non zero maximal ideal of R is cancellation ideal .

## Proof:-

Let $I$be a nonzero maximal ideal of R ,such that $AI=BI$ ,where A ,B are two ideal of R .

Now ,we have $AIM = BIM$ ,then $IAM = IBM$But M is max-fully cancellation module, therefore AM=BM

and also ,M is cancellation module , then A=B.

which is what we wanted.

However ,we have the following result **.**

## Proposition(1.11):-

Let M ,N be two R-module .If M$\cong N$ ,then M is max-fully cancellation module if and only if N is max-fully cancellation module.

## Proof:-

Let $\theta: M \longrightarrow N$ be an isomorphism . Suppose M is max-fully cancellation module .

To prove N is max-fully cancellation module.

For every non zero maximal ideal $I$ of R and every submodules$\grave{N}_1$ ,$\grave{N}_2$ of N .

**Let $I\grave{N}_1$=$I\grave{N}_2$ .**

Now , there exists two submodules $N_1$ ,$N_2$ of M such that

$\theta(N_1)=\grave{N}_1$ , $\theta(N_2)=\grave{N}_2$ .

Then $I\,\theta(N_1)= I\,\theta(N_2)$ , Which implies $\theta(I\,N_1)= \theta(I\,N_2)$

Therefore $IN_1 = IN_2$ (since $\theta$ is (1-1)).

But M is max-fully cancellation R-module . Then $N_1=N_2$ and hence $\theta(N_1)= \theta(N_2)$ .

Therefore $\grave{N}_1 =\grave{N}_2$ .

That is N is max –fully cancellation R-module .

## Conversely:

Suppose that N is max- fully cancellation R-module .

Let $IN_1 = IN_2$ for every non zero maximal ideal $I$ of R and every submodules $N_1$ , $N_2$ of M .

Now , $\theta(I\,N_1)= \theta(I\,N_2)$ .

Which implies $I\,\theta(N_1)= I\,\theta(N_2)$ ,where $\theta(N_1)$ , $\theta(N_2)$ are two submodule of N .

Also N is max-fully cancellation module. Then $\theta(N_1)= \theta(N_2)$

Which implies $N_1=N_2$ (since $\theta$ is (1-1))

Which completes the proof .

## Proposition(1.12):-

Let R be a principle ideal ring and M be an R-module such that ann($I$)=0 for each non zero ideal $I$ of R .Then M is max-fully cancellation module .

## Proof:-

Let $I$ be a non zero maximal ideal of R and $N_1$ ,$N_2$are submodules of M such that $IN_1 = IN_2$ .

By assumption I=(x) ,for some x≠o ,x∈ $R$.

Therefore $(x)N_1 =(x)N_2$ .To prove $N_1=N_2$ .

Let a∈$N_1$ .Then xa∈$(x)N_1=(x)N_2$ and hence xa=xb , for some b∈$N_2$.

Which implies that x(a-b)=0 and hence a-b∈ann($I$) =0.

Therefore a-b=0. Thus a=b and hence $N_1=N_2$.

Which implies , M is max-fully cancellation module .

The converse of proposition (1.12) is not true in general

**,for examples**

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The $Z_6$ –module $Z_6$ is max-fully cancellation module by remarks and examples (1.7) ,since $Z_6$ is principle ideal ring and $(\bar{2})$ is an ideal of $Z_6$ ,but ann$(\bar{2})\neq 0$.

The following lemma is needed in our next proposition

## Lemma(1.13):-

Let $R$ be any ring . $I$ be a proper ideal of R such that ann(M)$\subseteq I$ .If $I$ is maximal ideal of R ,then $\frac{I}{ann\,(M)}$ is maximal ideal of $\frac{R}{ann\,(M)}$ .

## Proof:-

Suppose that $I$ is maximal ideal of R .

We want To prove that $\frac{I}{ann\,(M)}$ is maximal ideal in $\frac{R}{ann\,(M)}$.

Assume that there exists an ideal $\frac{J}{ann\,(M)}$ of $\frac{R}{ann\,(M)}$ such that

$\frac{I}{ann\,(M)} \subsetneq \frac{J}{ann\,(M)}$ .

Then there exists x+ann(M)$\in \frac{J}{ann\,(M)}$and x+ann(M)$\notin \frac{I}{ann\,(M)}$ which implies x$\notin I$. But $I$ maximal ideal of R and x$\notin I$ , then R=( $I$,X).

Therefore 1=a+rx where a$\in I$,r$\in R$,

Hence$\theta(1)=\theta(a)+\theta(rx)$. where $\theta: R \longrightarrow R/ann(M)$ natural homomorphism .

Then 1+ann(M)=(a+ann(M))+(r+ann(M))(x+ann(M)).

Thus 1+ann(M)$\in \frac{J}{ann\,(M)}$ and hence $\frac{J}{ann\,(M)}=\frac{R}{ann\,(M)}$.

Therefore $\frac{I}{ann\,(M)}$ is maximal ideal of $\frac{R}{ann\,(M)}$ .

**Conversely:-** To prove that $I$ is maximal ideal in R

Suppose that there exists an ideal $J$ of R such that $I \subsetneq J$.

Then there exists x$\in J$ ,x$\notin I$ which implies x+ann(M)$\notin \frac{I}{ann\,(M)}$ .But $\frac{I}{ann\,(M)}$ is maximal ideal in $\frac{R}{ann\,(M)}$, then $\frac{R}{ann\,(M)}$=($\frac{I}{ann\,(M)}$ ,x+ann(M) ).Therefore 1+ann(M) =$\bar{m}$+(r+ann(M))(x+ann(M)) ,where $\bar{m} \in \frac{I}{ann\,(M)}$ ,$\bar{m}$=a+ann(M) and a$\in I$ .

1+ann(M)=(a+ann(M))+(rx+ann(M))

1+ann(M)=(a+rx)+ann(M) which implies that 1-(a+rx)$\in ann(M) \subseteq I$.

Then 1-(a+rx)$\in I$.

Then 1-a-rx=n ,n$\in I$.

Thus 1=n+a+rx $\in J$ .

Therefore $J=R$ which completes proof .

## Proposition(1.14):-

M is max-fully cancellation R-module if and only if M is max-fully cancellation $\bar{R}=\frac{R}{ann\,(M)}$ -module .

## Proof:-

($\Longrightarrow$) let M be a max –fully cancellation R-module .

Let $I$ be a non zero maximal ideal of $\bar{R}=\frac{R}{ann\,(M)}$ ,and $N_1$ ,$N_2$ are two $\bar{R}$-submodules.

Then $I$=$\frac{\acute{I}}{ann\,(M)}$ ,for some ann(M)$\subseteq \acute{I}$ and $N_1$ ,$N_2$ are R-submodules .

Now ,suppose $IN_1 = IN_2$ and we have for any x$\in \acute{I}$,x+ann(M)$\in I$,then (x+ann(M))=x$n \in \acute{I}n$,for every n$\in N_1$ .

But (x+ann(M))$N_1 \in IN_1$= $IN_2$ ,where x+ann(M)$\in I$.

Thus xn$\in IN_2$ ,then x$n_1$ =$\sum_{i=1}^{m} \bar{a}_i y_i$ where $\bar{a}_i \in I$ , $y_i \in N_2$ .

But for every i , 1≤i≤m , $\bar{a}_i = a_i$+ann(M) and hence xn=$\sum_{i=1}^{m}(a_i + ann(M))\, y_i$=$\sum_{i=1}^{m} a_i y_i \in \acute{I}N_2$ .Therefore $\acute{I}N_1 \subseteq \acute{I}N_2$.similarly$\acute{I}N_2 \subseteq \acute{I}N_1$ ,thus $\acute{I}N_1 = \acute{I}N_2$and since $\acute{I}$ is maximal ideal of R by lemma(1.13) ,also M is max-fully cancellation R- module.

Then $N_1=N_2$ and hence M is max-fully cancellation $\bar{R}$-module .

(⟸) The proof is similarly

## §2 Direct Sum Of Max-Fully cancellation Modules

In this section , we discuss the direct sum of max-fully cancellation modules and show that the direct sum of max-fully cancellation R-module needs not to be max-fully cancellation .However ,we give some conditions under which the class of max-fully cancellation modules is closed under direct sum **.**

### Definition(2.1)[6]:-

A submodule $M_1$ of M is **a direct summand** of M in case there is a submodule $M_2$ of M with $M=M_1 \oplus M_2$ .

The following proposition proves that the direct summand of max-fully cancellation is also max-fully cancellation under the condition ann$M_1$+ann$M_2$=R .

### Proposition(2.2):-

Let $M=M_1 \oplus M_2$ be an R-module ,where $M_1$ ,$M_2$ are two submodules of M such that ann$M_1$+ann$M_2$=R Then $M_1$ and $M_2$ are max-fully cancellation R-modules if and only if M is max-fully cancellation.

**Proof:-** (⟹)To prov**e** M is max-fully cancellation .Let $I$ be a non zero maximal ideal of R and $N_1$ ,$N_2$ are two submodules of M such that$IN_1=IN_2$.

Since ann$M_1$+ann$M_2$=R then by the [12] we get $N_1=A_1+A_2$ and $N_2=B_1+B_2$ for some,$A_1$,$B_1$ are submodule of $M_1$ and$A_2$,$B_2$ are submodules of $M_2$.

Thus $I(A_1+A_2)= I(B_1+B_2)$ .

Then $IA_1+IA_2 =IB_1+IB_2$ .

Which implies $IA_1=IB_1$ and $IA_2=IB_2$.

But $M_1$ ,$M_2$ are max-fully cancellation R-module.

Then $A_1=B_1$ and $A_2=B_2$ ,Thus $N_1=N_2$ .

(⟸)

since $M_1 \subseteq M=M_1 \oplus M_2$ ,but M is max-fully cancellation Then $M_1$ is max-fully cancellation.

And $M_2 \subseteq M$ ,then $M_2$ is max-fully cancellation

### Definition(2.3)[15]:-

A submodule N of an R-module M is called **invariant** if f(N) ⊆ $N$ for each f∈ $END_R$(M)

### Definition(2.4)[12]:-

An R-module M is called **fully invariant** if every submodule of M is an invariant .

### Remark (2.5)[12]:-

Every invariant R-module is fully invariant and the converse is not true in general.

### Remark (2.6):

Every submodule of invariant module is invariant.

The following proposition also shows that the direct sum of max-fully cancellation modules is also max-fully cancellation ,under another condition ann$M_1$+ ann$M_2$=R.

### Proposition(2.7):-

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Let $M=M_1 \oplus M_2$ be an R-module where $M_1$ ,$M_2$ are two submodules of M such that $M_1$,$M_2$ are fully invariant submodules . Then $M_1$,$M_2$ are max-fully cancellation R-modules if and only if M is max-fully cancellation R-module .

## Proof-:

($\Longrightarrow$) suppose that $M_1$ , $M_2$ are max-fully cancellation .

Now , let $N_1$ ,$N_2$ are submodules of M and let $I$ be a non zero maximal ideal of R .

Suppose $IN_1=IN_2$ since $M_1$ ,$M_2$ are fully invariant submodule

Then $N_1=(N_1\cap M_1)\oplus(N_1\cap M_2)$ and $N_2=(N_2\cap M_1)\oplus(N_2\cap M_2)$ [26].

Therefore $I\ ((N_1\cap M_1)\oplus(N_1\cap M_2)) = I((N_2\cap M_1)\oplus(N_2\cap M_2))$.

So $I(N_1\cap M_1) = I(N_2\cap M_1)$ and $I(N_1\cap M_2) = I(N_2\cap M_2)$ .

Then $N_1\cap M_1= N_2\cap M_1$ and $N_1\cap M_2 = N_2\cap M_2$ since $M_1$ , $M_2$ are max-fully cancellation .

Then $N_1=N_2$ .

($\Leftarrow$)suppose that M is max-fully cancellation module .

Since $M_1\subseteq M=M_1\oplus M_2$ and $M_2\subseteq M=M_1\oplus M_2$

But M is max-fully cancellation then by remarks and examples (1.7),we get , $M_1$ and $M_2$ are max fully cancellation module .

## Proposition(2.8):-

Let $M_1$ ,$M_2$ be two R-modules and $P_1$ ,$P_2$ are two submodules of $M_1$ ,$M_2$ respectively such that $annM_1+annM_2=R$ Then $P_1$ ,$P_2$ are max-fully cancellation R-module if and only if $P_1\oplus P_2$ is max-fully cancellation R-module of $M_1\oplus M_2$.

## Proof:-

**($\Rightarrow$) F**or each non zero maximal ideal $I$ of R and $K_1\oplus W_1$ ,$K_2\oplus W_2$ are submodules of $P_1\oplus P_2$ .

Suppose$I(K_1\oplus W_1)= I(K_2\oplus W_2)$

Then$IK_1\oplus IW_1= IK_2\oplus IW_2$ .

Which implies $IK_1= IK_2$ and $IW_1= IW_2$ . But $P_1$ ,$P_2$ are max-fully cancellation R-modules

Then $K_1=K_2$ and $W_1=W_2$ ,hence $K_1\oplus W_1= K_2\oplus W_2$ .

($\Leftarrow$) since $P_1\subseteq P_1\oplus P_2$ , $P_2\subseteq P_1\oplus P_2$

The result follows from Remark (1.7).

## Remark (2.9):-

A direct summand of R-module which is max-fully cancellation is also max-fully cancellation .

## Proof:-

It is obvious from remark and examples(1.7 ).

### Remark(2.10):-

The converse of remark(2.9) is not true in general

**for example :** The Z-module $M=Z\oplus Z$ is not max-fully cancellation Z-module , since $(2)(Z\oplus(0))=(2)((0)\oplus Z)=2Z$ ,where (2) is maximal ideal of $Z$ and $(Z\oplus(0))$ , $((0)\oplus Z)$ are two submodules of M .

But $(0)\oplus Z \neq Z\oplus(0)$,while$Z$as a $Z$-module is max-fully cancellation by remark and examples (1.7).

From remark (2.10) ,we obtain the following

### Remark(2.11):

$It$ is not necessary that $M^2=M\oplus M$ is max-fully cancellation module if M is max-fully cancellation R- module.

## Definition (2.12)[6]:-.

A ring R is said to be **chained ring** if every non- empty set of ideals in R with respect to inclusion as ordering .

The following result is an immediate consequence of remark (2.9)

## Corollary (2.13):-

Let R be a chained ring .Then ,the direct summand of two max-fully cancellation R-module is also a max-fully cancellation R-module.

## §3 max-fully cancellation modules and

A subset S of a ring R is called multiplicatively closed if 1∈S and ab∈S for every a , b∈ S. we know that every proper ideal P in R is prime if and only if R-P is multiplicatively closed[11]

Let M be a module on the ring R and S be a multiplicatively closed on R such that S≠0 and let $R_s$ be the set of all fractional $\frac{r}{s}$ where r∈R and s∈S and $M_s$ be the set of all fractional $\frac{x}{s}$ where x∈M , s∈S. For $x_1, x_2$ ∈M and $s_1, s_2$ ∈S , $\frac{x_1}{s_1}$ =$\frac{x_2}{s_2}$ if and only if there exist t∈S such that t($s_1 x_2 - s_2 x_1$) =0.

So , we can make $M_s$ into $R_s$ -module by setting $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$, $\frac{r}{t} . \frac{x}{s} = \frac{rx}{ts}$ , for every x,y∈M and every r∈R , s,t∈S.

If S=R-P where P is a prime ideal we used $M_p$ instead of $M_s$ and $R_p$ instead of $R_s$ . A ring in which there is only one maximal ideal is called a local ring . Hence , $R_p$ is often called the localization of R at P , similar $M_p$ is the localization of M at P. so we can define the two maps $\psi: R \longrightarrow R_s$ such that $\psi(r)$ =r/ 1 , ∀r∈R , $\theta: M \longrightarrow M_s$ , such that $\theta(m)$ =m/1 , ∀m∈M.

Recall that if N be a submodule of an R-module M and S be a multiplicatively closed in R so $N_s$ ={$\frac{n}{s}$ : n∈N,s∈ S} be submodule on $R_s$ –module $M_s$ , see [11].

In this section we study the behavior of max-fully cancellation R-module under localization and many results are provided .

### Definition(3.1)[13]:-

Let M be an R-module .For all submodules N of M we shall denote the extension N in $M_p$ by $N^e$ and for all submodules L in $M_p$ we shall denote the contraction of L in $M_p$ by $L^c$ and $L^c$ means $f^{-1}(L)$ ;where f:M→ $M_p$ is the natural homomorphism

For our next proposition ,the following lemma is needed.

### Lemma(3.2):-

Let R be a ring and let $I$ be an ideal of R.Then $I$ is maximal ideal of R if and only if $I_p$ is maximal ideal of $R_p$ ,for every maximal ideal P of R.

### Proof:-

Suppose that $I$ is maximal ideal of R . Let $J_p$ be an ideal of $R_p$ such that $I_p \subsetneq J_p$,then there exists $\frac{as}{s} \in J_p$ , $\frac{as}{s} \notin I$, Therefore a$\notin I$

,but $I$ is maximal in R, then $(I, a)$=R and hence x+ra=1 for some r∈R,x∈ $I$.

Then $\frac{xs^2}{s^2}$+$\frac{rs}{s}$ .$\frac{as}{s}$=$1_p \in J_p = R_p$.

Which implies $I_p$ maximal ideal in $R_p$.

Now, suppose that $I_p$ is maximal ideal in $R_p$

Let $J$ be an ideal of R such that $I \subsetneq J$,

Then there exists x∈ $J$,x$\notin I$.

Which implies $\frac{xs}{s} \notin I_p$ ,but $I_p$ is maximal ideal in $R_p$ ,and $\frac{xs}{s} \notin I_p$.

Then ($I_p$ ,$\frac{xs}{s}$)=$1_p$ and hence $\frac{as^2}{s^2}$+$\frac{rs}{s}$.$\frac{xs}{s}$=$1_p$.

Therefore $\frac{as^2}{s^2}$+$\frac{rxs^2}{s^2}$=$1_p$ which implies a+rx=1∈ $J = R$

Then $I$ is maximal ideal in $R$.

## Lemma(3.4)[10]:-

Let M be an R-module ,and let A ,B are submodule of M Then A=B if and only if $A_p$=$B_p$ ,for every maximal ideal of R.

The following proposition shows that the concept of max-fully cancellation modules is equivalent between a module M and locally of M.

## Proposition(3.5):-

Let M be R-module then $M_p$ is max- fully cancellation (for every maximal ideal P of R) if and only if M is max-fully cancellation R-module.

**Proof:-**

Suppose that $IN$= $IK$ where $I$ is anon zero maximal ideal of R and N ,K are any two submodules of M.

Then $(IN)_p$=$(IK)_p$ for every maximal ideal P of R by lemma(3.4).

Then $I_p N_p$= $I_p K_p$ [16]

But $M_p$ is max-fully cancellation so $N_p$=$K_p$ for every maximal ideal P of R.

Thus by lemma(3.4) we have N=K.

## Conversely:

Let P be any maximal ideal , $I$ bean maximal ideal of R and let A be a submodule of M ,

We have $I_p \frac{a}{s} \in I_p B_p$ ,where $I_p$ is an maximal ideal of the ring $R_p$ and ,$A_p B_p$ are submodules of $R_p$ –module $M_p$ and $\frac{a}{s} \in A_p$.

Thus for any x∈ $I$ we have $\frac{x}{1} \in I_p$ and $\frac{x}{1}.\frac{a}{s} \in I_p$ . $B_p$ and then

$\frac{xa}{s}$=$\sum_{i=1}^{n} \frac{K_i}{S_i}.\frac{b_i}{t_i}$ where $k_i \in I$,$b_i \in B$,$S_i$ ,$t_i \notin P$.

Thus $\frac{xa}{s}$= $\sum_{i=1}^{n} \frac{k_i b_i}{\bar{s}_i}$ where $\bar{s}_i$=$s_i t_i$

Therefore $\frac{xa}{s}$=$\frac{k_1 b_1 u_1+k_2 b_2 u_2+\cdots\ldots\ldots\ldots+k_n b_n u_n}{v}$ .

Where v=$\bar{s_1}\bar{s_2}\bar{s_3} \ldots\ldots\ldots\bar{s_n}$ ,$u_1$ =$\bar{s_1}\bar{s_3} \ldots\ldots\ldots\bar{s_n}$ ,$u_n$ =$\bar{s}_1\bar{s}_2 \ldots \bar{s}_{n\_1}$

Thus there exists K∉ $P$ such that Kxav=($k_1b_1u_1$+$k_2b_2u_2$+…….+$k_nb_nu_n$)$S_k$ But kxav∈ $I_A$ ,

($k_1b_1u_1$+$k_2b_2u_2$+……..+$k_nb_nu_n$)$S_k$∈ $IB$

But M is max-fully cancellation so by [5] we have a∈ $B$ Thus $\frac{a}{s} \in B_p$

Therefore $M_p$ is max-fully cancellation $R_p$-module.

Now ,we have the following proposition.

## Proposition(3.6):-

Let M be an R-module and N,L be two finitely generated submodules of M .if $N_p$,$L_p$ are max-fully cancellation then N∩ $L$ is max-fully cancellation R-submodule.

## Proof:-

Let N and L be two finitely generated submodules of M .

Then $(N_p$:$L_p)$+$(L_p$+$N_p)$=$R_p$ for all maximal ideal P of R[15]

Therefore $L_p \cap N_p$=$N_p$ or $N_p \cap L_p$ $= L_p$ .

Which implies $L_p \cap N_p$ is max-fully cancellation , but $L_p \cap N_p$=$(L\cap N)_p$

then $(L\cap N)_p$ is max-fully cancellation and L∩ $N$ is max –fully cancellation R-submodule by(3.5).

## Proposition(3.7):-

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Let M be an R-module and N,L be two finitely generated submodules of M .if $N_p$ ,$L_p$ are max-fully cancellation$R_p$-module then N+$L$ is max-fully cancellation R-module.

**Proof:-**

Let N ,L be two finitely generated submodules of M.

Then ($N_p$:$L_p$)+($L_p$:$N_p$)=$R_P$ for all maximal ideal P of R[16].

Now ,let $r_1 \in (N_p:L_p)$ and $r_2 \in (L_p:N_p)$ such that $r_1+r_2=1$ ,

Then either $r_1$ is a unite element or $r_2$ is a unite element (since $R_p$ is local ring)

Which implies ($N_p$:$L_p$)=$R_p$ or ($L_p$:$N_p$)=$R_p$ ,

Thus either$L_p \subseteq N_p$ or $N_p \subseteq L_p$ .

Then $L_p$+$N_p$=$N_p$ or $N_p$+$L_p$=$L_p$

Therefore $N_p$+$L_p$ is max-fully cancellation $R_p$-submodule(since $N_p$ and $L_p$ are max-fully cancellation $R_p$-submodules).

Which implies ,$(L+N)_p$ is max-fully cancellation $R_p$-submodule and hence by (3.5),we get L+N is max- fully cancellation R-submodule.

## §4 The relationship between Max-fully cancellation Modules and its trace

In this section we give some relationships between the modules having the max-fully cancellation module property and its trace , see proposition (4.4) , proposition(4.7) and proposition(4.8).

### "Definition (4.1)[1]:

The **Dual** of M denote by $M^*$ and defined by $M^*$=$HOM_R$( M,R) and the **dibual** of M denoted by $M^{**}$ is defined $M^{**}$=$HOM_R$($M^*$,R)."

### "Definition(4.2)[8]:-

The **trace** of an R-module M denoted by T(M) is defined byT(M) =$\sum_{i\in\wedge} \theta_i(M)$ where the sunnruns over all $\theta_i$ In $HOM_R$ (M,R)."

### "Definition (4.3)[12]:-

An R-module M is said to be a **multiplication module** if for every submodule N of M there exists an ideal $I$ of R such that N= $I$M ".

Now ,we state and prove the following result .

### Proposition(4.4):

Let M,N be two R-modules such that M is multiplication R-modules and let L=$\sum_{finite} \psi_\lambda(M)$ be a cancellation submodule of N ,where the sum is taken as a subset of

$HOM_R$(M,N).Then M is max-fully cancellation R-module when L is fully cancellation submodule.

**Proof:-**

Let $I$$N_1$=$I$$N_2$ ,for every a non zero maximal ideal of R and $N_1$,$N_2$ be any two submodules of M .

Now ,there exists two ideals A,B of R such that $N_1$=AM,$N_2$=BM(since M is multiplication).

Then $I$AM= $I$BM and next$\psi_\lambda(IAM)$=$\psi_\lambda$(IBM) and hence $\sum_{finite} \psi_\lambda(IAM)$=$\sum_{finite} \psi_\lambda(IBM)$,which implies that

$I$A$\sum_{finite} \psi_\lambda(M) = I$B$\sum_{finite} \psi_\lambda(M)$.

Therefore $I$AL =$I$BL and hence A=B (since L is fully cancellation submodule and also,L is cancellation submodule)

Thus AM=BM and hence $N_1$=$N_2$

Which completes the proof.

## Definition(4.5)[11]:-

A fractional ideal A of a ring R is **invertible** if there exists a fractional ideal B of R such that AB=R. where A fractional ideal of a ring R is a subset A of the total quotient ring K of R such that

(1)A is an R-module , that is , if a,b$\in A$ and r$\in R$,then a-b, ra$\in A$;and

(2)there exists a regular element d of R such that dA$\subseteq R$.

## Remark(4.6)[5]:-

An invertible ideal is a cancellation ideal.

The following corollaries is an immediately proposition (4.4)

## Corollary(4.7):

let M be a multiplication R-module and T(M)is an invertible and fully-cancellation ideal of R. then M is max –fully cancellation module.

## Proof:-

Directly from the definition of T(M).]and by remark (4.6) and by proposition (4.4).

## Corollary(4.8):

let M be a multiplication R-module and N be a cancellation homomorphic image of M . If N is fully cancellation submodule ,then M is max-fully cancellation module

### proof:-

Let $I$ be a non-zero maximal ideal of R and $N_1$,$N_2$ are two submodules of M such that $IN_1=IN_2$ and $\theta$(M)=N

Then $N_1$=AM,$N_2$=BM for some ideals A,B of M.

Therefore $IAM= IBM$ and hence $\theta(IAM)= \theta(IBM)$.

and next $IA\theta(M)= IB\theta(M)$.

Which implies that $I$AN=$I$BN .But N is fully cancellation and cancellation module.

Then A=B and hence AM=BM . finally we get $N_1=N_2$.

## Definition(4.9)[7]:-

An R-module M called **projective** if for every R-epimorphisim h:A$\longrightarrow B$ and f$\in HOM_R$ (M,B) ,there exists g$\in HOM_R$(M,B) such that h∘g=f.

The following proposition gives a sufficient conditions for a module M to be max –fully cancellation.

## Proposition(4.10):-

Let M be a multiplication projective R-module and T(M) is fully cancellation ideal .Then M is max-fully cancellation module.

## .Proof:

let $I$ be a nonzero maximal ideal $I$of R and $N_1$,$N_2$be two submodules of M such that

$IN_1=IN_2$

Let $N_1$=AM ,$N_2$=BM for some ideals A and B of R (since M is multiplication)

Now , $I$AM=$I$BM.

Then $\theta_i(I\text{AM})=IA\ \theta_i$(M)=$\theta_i(IBM)$= , $IB\ \theta_i$(M)

And hence $I\text{A}\sum_{i\in\wedge}\theta_i$(M)= $I\text{B}\sum_{i\in\wedge}\theta_i$(M)

Which implies $I$AT(M)= $I$BT(M). but T(M) is fully cancellation

Then AT(M)=BT(M) ,we have M is projective ,then T(M)M=M

And hence AT(M)M=BT(M)M.

Therefore AM=BM and hence $N_1=N_2$.

The following proposition gives a characterization for max-fully cancellation module.

## Proposition(4.11)

Let M be a cancellation R-module and Ker$\sum_{i=1}^{n}\theta_i$(M)=0,where$\theta_i$ is taken as a subset of $Hom_R$(M,R). Then the following are equivalents:

**(1)**M is max-fully cancellation module.

**(2)**T(M) is max-fully cancellation ideal.

## Proof:-

(1)⟹(2) :Assume that M is max-fully cancellation module .

To prove that T(M) is max-fully cancellation ideal for every a non zero maximal ideal $I$ of R and two an ideals A and B of T(M).

Let $IA=IB$ .Then $IAM=IBM$. but M is max-fully cancellation module and AM ,BM are submodule of M.

Therefore AM=BM and hence A=B (since M is cancellation module)

Therefore T(M) is max-fully cancellation ideal .

(2)⟹(1) :Assume that T(M) is max-fully cancellation ideal

To show that M is max-fully cancellation module.

Let for every a non zero maximal ideal $I$ of R and any two submodules W ,K of M such that $IW= IK$.

Now$\theta_i(IW)$=$\theta_i(I$K) and next $\sum_{i=1}^{n}\theta_i(IW)$=$\sum_{i=1}^{n}\theta_i(IK)$ But $\theta_i(IW)$=$I\ \theta_i(W)$=$\theta_i(IK)$=$I\ \theta_i$(K).

Therefore $I\sum_{i=1}^{n}\theta_i(W)$=$I\ \sum_{i=1}^{n}\theta_i(K)$ and hence $I$ T(W)= $I$ T(K).

But T(W) ,T(K) are subideals of T(M) and T(M) is max-fully cancellation ideal ,then T(W)=T(K) .

To prove W=K . let $w_i$ ∈W. then$\theta_i(w_i) \in \theta_i(W)$.

$\sum_{i=1}^{n}\theta_i(w_i) \in \sum_{i=1}^{n}\theta_i(W)$=T(W)=T(K) And hence $\sum_{i=1}^{n}\theta_i(w_i)$ ∈T(K) = $\sum_{i=1}^{n}\theta_i(K)$.

Therefore $\sum_{i=1}^{n}\theta_i(w_i)$= $\sum_{i=1}^{n}\theta_i(k_i)$ And hence $\sum_{i=1}^{n}\theta_i(w_i-k_i)$=0

Which implies ,$w_i-k_i \in \mathrm{Ker}\sum_{i=1}^{n}\theta_i$=0 .

Then $w_i-k_i$=0 and hence$w_i$=$k_i$ .

Thus W ⊆ K ,similarly we can show that K ⊆ W

And hence W= K. This end the proof .

Next ,we have the following proposition

## Proposition(4.12):-

Let M be an R-module .M is max-fully cancellation module ,if T(M) is fully cancellation ideal such that $\sum\varphi_\lambda(M)$=0 ,where$\varphi_\lambda \in \mathrm{Hom}(M,R)$.

## Proof:

By the same way of the second side of proof of proposition (4.11) by using T(M) is fully cancellation instead of T(M) is max-fully cancellation .

Now ,we end this section by the following proposition

## Proposition(4.13):

Let M be a cancellation R-module .T(M) is max-fully cancellation ideal ,if M is fully cancellation module .

## Proof:-

By the steps of the first side of proof of proposition (4.11) and we take M is fully cancellation instead of M is max-fully cancellation module.

## Reference


[1]C. Faith. ,Algebra Rings "Modules and categories I", Springer-Verlag , Berlin , New York , 1973.

[2]C .P. Lu ,"Prime submodules of Modules", commtent Math.,Unniv . Stpaul,33(1984),61-69.

[3]D.D Anderson and D.F Anderson , "Some remarks on cancellation ideals ", math. Japonica 29(6),(1984), 879-886

[4]D.M. Burton., "Abstract and Linear Algebra", Univ.of New Hamphire , (1971).

[5]D. Hadwin. And J.W. Kerr., "Scalar reflexive ring", Amer. Math. Soc.,103(1),May (1988),1-8.

[6]F.W. Anderson and K.R. Fuller. "Rings and Categories of Modules", Univ. of Oregon ,1973.

[7]F. Kash ," Modules and Rings", Academic Press, London ,1982.

[8]F.W Anderson and K.R. Fuller."Rings and Categories of Modules", Springer-Verlage, Berline

, Heidera New York ,(1974).

[9]H.Y .Khalaf," semimaximal submodules" Ph.D. Thesis , Univ. of Baghdad,(2007).

[10]I.M.A. Hadi and A .A .Elewi ,"fully cancellation and naturaly cancellation modules" Journal of Al-Nahrain Univ.,17(3),September(2014),pp.178-184.

[11] M.D .Larsen and Maccar the P .J., "Multiplication Theory of ideal", Academic Press, London ,New York,(1971).

[12] M. S. Abbas, "On fully stable modules", Ph .D. Thesis , Univ. of Baghdad,(1990).

[13]M.F. Atriyah and I . Gmacdonald , "Introduction to Commutative Algebra", Univ . of Oxford , (1969).

[14]R.W .Gilmer, "Multiplicative ideal Theory", Marcel Dekker, New York ,(1972)

[15] علي سبع مجباس ,(حول موديولات الحذف) , رسالة ماجستير , كلية العلوم ,جامعة بغداد(1990)

[16] نهاد سالم عبد الكريم ,(حول موديولات المنتظمة من النمط(Z)) , رسالة ماجستير , كلية العلوم ,جامعة بغداد(1993)